

ANOVA (analysis of variance) in the quantum linguistic formulation of statistics

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Abstract

Recently, we proposed quantum language (or, measurement theory), which is characterized as the linguistic turn of the Copenhagen interpretation of quantum mechanics. We believe that this language has a great power of description, and therefore, even statistics can be described by quantum language. In this paper, we show that ANOVA (analysis of variance (one-way and two-way)) can be formulated in quantum language. Since quantum language is suited for theoretical arguments, we believe that our results are visible and understandable. For example, we can answer the question "What kind of role does Kolmogorov's probability theory play in ANOVA?" That is, the readers find that Kolmogorov's probability theory is merely used in order to calculate multi-dimensional Gauss integrals, and thus, they can avoid to confuse the relation between Kolmogorov's probability theory and statistics.

(Key words: Quantum language, Statistical hypothesis testing, ANOVA, F-distribution, Student's t-distribution, Chi-squared distribution,)

1 Introduction

1.1 Quantum language (Axioms and Interpretation)

As mentioned in the above abstract, our purpose is to understand ANOVA (analysis of variance) in terms of quantum language, which is proposed in [2]- [9].

According to ref. [9], we shall mention the overview of quantum language (or, measurement theory, in short, MT).

Quantum language is characterized as the linguistic turn of the Copenhagen interpretation of quantum mechanics (cf. refs. [5], [11]). Quantum language (or, measurement theory) has two simple rules (i.e. Axiom 1 (concerning measurement) and Axiom 2 (concerning causal relation)) and the linguistic interpretation (= how to use the Axioms 1 and 2). That is,

$$\begin{array}{ccccccc} \boxed{\text{Quantum language}} & = & \boxed{\text{Axiom 1}} & + & \boxed{\text{Axiom 2}} & + & \boxed{\text{linguistic interpretation}} \\ (= \text{MT (measurement theory)}) & & (\text{measurement}) & & (\text{causality}) & & (\text{how to use Axioms}) \end{array} \quad (1)$$

(cf. refs. [2]- [9]).

This theory is formulated in a certain C^* -algebra \mathcal{A} (cf. ref. [12]), and is classified as follows:

$$(A) \quad \text{MT} \begin{cases} \text{quantum MT} & (\text{when } \mathcal{A} \text{ is non-commutative}) \\ \text{classical MT} & (\text{when } \mathcal{A} \text{ is commutative, i.e., } \mathcal{A} = C_0(\Omega)) \end{cases}$$

where $C_0(\Omega)$ is the C^* -algebra composed of all continuous complex-valued functions vanishing at infinity on a locally compact Hausdorff space Ω .

Since our concern in this paper is concentrated to the usual statistical hypothesis test methods in statistics, we devote ourselves to the commutative C^* -algebra $C_0(\Omega)$, which is quite elementary. Therefore, we

believe that all statisticians can understand our assertion (i.e., a new viewpoint of the confidence interval methods).

Let Ω is a locally compact Hausdorff space, which is also called a state space. And thus, an element $\omega(\in \Omega)$ is said to be a state. Let $C(\Omega)$ be the C^* -algebra composed of all bounded continuous complex-valued functions on a locally compact Hausdorff space Ω . The norm $\|\cdot\|_{C(\Omega)}$ is usual, i.e., $\|f\|_{C(\Omega)} = \sup_{\omega \in \Omega} |f(\omega)|$ ($\forall f \in C(\Omega)$).

Motivated by Davies' idea (cf. ref. [1]) in quantum mechanics, an observable $O = (X, \mathcal{F}, F)$ in $C_0(\Omega)$ (or, precisely, in $C(\Omega)$) is defined as follows:

(B₁) X is a topological space. $\mathcal{F}(\subseteq 2^X$ (i.e., the power set of X) is a field, that is, it satisfies the following conditions (i)–(iii): (i): $\emptyset \in \mathcal{F}$, (ii): $\Xi \in \mathcal{F} \implies X \setminus \Xi \in \mathcal{F}$, (iii): $\Xi_1, \Xi_2, \dots, \Xi_n \in \mathcal{F} \implies \cup_{k=1}^n \Xi_k \in \mathcal{F}$.

(B₂) The map $F : \mathcal{F} \rightarrow C(\Omega)$ satisfies that

$$0 \leq [F(\Xi)](\omega) \leq 1, \quad [F(X)](\omega) = 1 \quad (\forall \omega \in \Omega)$$

and moreover, if

$$\Xi_1, \Xi_2, \dots, \Xi_n, \dots \in \mathcal{F}, \quad \Xi_m \cap \Xi_n = \emptyset \quad (m \neq n), \quad \Xi = \cup_{k=1}^{\infty} \Xi_k \in \mathcal{F},$$

then, it holds

$$[F(\Xi)](\omega) = \lim_{n \rightarrow \infty} \sum_{k=1}^n [F(\Xi_k)](\omega) \quad (\forall \omega \in \Omega)$$

Note that Hopf extension theorem (cf. ref. [13]) guarantees that $(X, \mathcal{F}, [F(\cdot)](\omega))$ is regarded as the mathematical probability space.

Now we shall briefly explain "quantum language (1)" in classical systems as follows: A measurement of an observable $O = (X, \mathcal{F}, F)$ for a system with a state $\omega(\in \Omega)$ is denoted by $M_{C_0(\Omega)}(O, S_{[\omega]})$. By the measurement, a measured value $x(\in X)$ is obtained as follows:

Axiom 1 (Measurement)

- The probability that a measured value $x(\in X)$ obtained by the measurement $M_{C_0(\Omega)}(O \equiv (X, \mathcal{F}, F), S_{[\omega_0]})$ belongs to a set $\Xi(\in \mathcal{F})$ is given by $[F(\Xi)](\omega_0)$.

Axiom 2 (Causality)

- The causality is represented by a Markov operator $\Phi_{21} : C_0(\Omega_2) \rightarrow C_0(\Omega_1)$. Particularly, the deterministic causality is represented by a continuous map $\pi_{12} : \Omega_1 \rightarrow \Omega_2$

Interpretation (Linguistic interpretation). Although there are several linguistic rules in quantum language, the following is the most important:

- Only one measurement is permitted.

In order to read this paper, it suffices to understand the above three (particularly, Axiom 1). For the further arguments, see refs. [2]- [9].

Example 1 [A kind of normal observable]. Let n be a natural number. Then, we get a kind of normal observable $O_G^n = (\mathbb{R}^n, \mathcal{B}_{\mathbb{R}}^n, G^n)$ in $C_0(\mathbb{R} \times \mathbb{R}_+)$. That is,

$$\begin{aligned} [G^n(\bigtimes_{k=1}^n \Xi_k)](\omega) &= \bigtimes_{k=1}^n [G(\Xi_k)](\omega) \\ &= \frac{1}{(\sqrt{2\pi}\sigma)^n} \int \cdots \int \exp\left[-\frac{\sum_{k=1}^n (x_k - \mu)^2}{2\sigma^2}\right] dx_1 dx_2 \cdots dx_n \\ &\quad \times_{k=1}^n \Xi_k \end{aligned} \quad (2)$$

$$(\forall \Xi_k \in \mathcal{B}_{\mathbb{R}} (= \text{Borel field in } \mathbb{R}), (k = 1, 2, \dots, n), \quad \forall \omega = (\mu, \sigma) \in \Omega = \mathbb{R} \times \mathbb{R}_+).$$

Fisher's maximum likelihood method (*cf.* refs. [3]- [9]) urges us to define the maps $\bar{\mu} : \mathbb{R}^n \rightarrow \mathbb{R}$, $\bar{\sigma} : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\bar{SS} : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\bar{\mu}(x) = \bar{\mu}(x_1, x_2, \dots, x_n) = \frac{x_1 + x_2 + \dots + x_n}{n} \quad (\forall x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n) \quad (3)$$

$$\bar{\sigma}(x) = \bar{\sigma}(x_1, x_2, \dots, x_n) = \sqrt{\frac{\sum_{k=1}^n (x_k - \bar{\mu}(x))^2}{n}} \quad (\forall x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n) \quad (4)$$

and

$$\bar{SS}(x) = \bar{SS}(x_1, x_2, \dots, x_n) = \sum_{k=1}^n (x_k - \bar{\mu}(x))^2 = n(\bar{\sigma}(x))^2 \quad (\forall x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n) \quad (5)$$

Thus, we have the following two image observables $\bar{\mu}(\mathcal{O}_G^n) = (\mathbb{R}, \mathcal{B}_{\mathbb{R}}, G^n \circ \bar{\mu}^{-1})$ and $\bar{SS}(\mathcal{O}_G^n) = (\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+}, G^n \circ \bar{SS}^{-1})$ in $C_0(\mathbb{R} \times \mathbb{R}_+)$, which are obtained by the formulas of Gauss integrals.

$$\begin{aligned} [(G^n \circ \bar{\mu}^{-1})(\Xi_1)](\omega) &= \frac{1}{(\sqrt{2\pi}\sigma)^n} \int \dots \int_{\{x \in \mathbb{R}^n : \bar{\mu}(x) \in \Xi_1\}} \exp\left[-\frac{\sum_{k=1}^n (x_k - \mu)^2}{2\sigma^2}\right] dx_1 dx_2 \dots dx_n \\ &= \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} \int_{\Xi_1} \exp\left[-\frac{n(x - \mu)^2}{2\sigma^2}\right] dx \end{aligned} \quad (6)$$

and

$$\begin{aligned} [(G^n \circ \bar{\sigma}^{-1})(\eta)](\omega) &= \frac{1}{(\sqrt{2\pi}\sigma)^n} \int \dots \int_{0 < \bar{\sigma}(x) \leq \eta} \exp\left[-\frac{\sum_{k=1}^n (x_k - \mu)^2}{2\sigma^2}\right] dx_1 dx_2 \dots dx_n \\ &= \int_0^{n\eta^2/\sigma^2} p_{n-1}^{\chi^2}(x) dx \\ & \quad (\forall \Xi_1 \in \mathcal{B}_{\mathbb{R}} (= \text{Borel field in } \mathbb{R}), \quad \forall \eta > 0, \quad \forall \omega = (\mu, \sigma) \in \Omega \equiv \mathbb{R} \times \mathbb{R}_+). \end{aligned} \quad (7)$$

Here, $p_{n-1}^{\chi^2}(x)$ is the chi-squared distribution with $n - 1$ degrees of freedom. That is,

$$p_{n-1}^{\chi^2}(x) = \frac{x^{(n-1)/2-1} e^{-x/2}}{2^{(n-1)/2} \Gamma((n-1)/2)} \quad (x > 0) \quad (8)$$

where Γ is the gamma function.

Remark 1 [The formulas of Gauss integrals]. Although the above (6) and (7) can be obtained by direct calculations, we consider that the calculation in the framework of Kolmogorov's probability theory (ref. [10]) is the most elegant. This kind of problem (i.e., the formulas of Gauss integrals) will be repeatedly discussed in this paper (*cf.* Remark 2 later).

1.2 The reverse relation between confidence interval and statistical hypothesis testing

Let $\mathcal{O} = (X, \mathcal{F}, F)$ be an observable formulated in a commutative C^* -algebra $C_0(\Omega)$. Let X be a topological space. Let Θ be a locally compact space with the semi-distance d_{Θ}^x ($\forall x \in X$), that is, for each $x \in X$, the map $d_{\Theta}^x : \Theta^2 \rightarrow [0, \infty)$ satisfies that (i): $d_{\Theta}^x(\theta, \theta) = 0$, (ii): $d_{\Theta}^x(\theta_1, \theta_2) = d_{\Theta}^x(\theta_2, \theta_1)$, (iii): $d_{\Theta}^x(\theta_1, \theta_3) \leq d_{\Theta}^x(\theta_1, \theta_2) + d_{\Theta}^x(\theta_2, \theta_3)$.

Let $E : X \rightarrow \Theta$ and $\pi : \Omega \rightarrow \Theta$ be continuous maps, which are respectively called an estimator and a quantity. Let α be a real number such that $0 < \alpha \ll 1$, for example, $\alpha = 0.05$. For any state $\omega (\in \Omega)$, define the positive number $\eta_\omega^\alpha (> 0)$ such that:

$$\eta_\omega^\alpha = \inf\{\eta > 0 : [F(\{x \in X : d_\Theta^x(E(x), \pi(\omega)) \geq \eta\})](\omega) \leq \alpha\} \\ \left(= \inf\{\eta > 0 : [F(\{x \in X : d_\Theta^x(E(x), \pi(\omega)) < \eta\})](\omega) \geq 1 - \alpha\} \right) \quad (9)$$

Then Axiom 1 says that:

(C₁) the probability, that the measured value x obtained by the measurement $M_{C_0(\Omega)}(\mathbf{O} := (X, \mathcal{F}, F), S_{[\omega_0]})$ satisfies the following condition (10), is more than or equal to $1 - \alpha$ (e.g., $1 - \alpha = 0.95$).

$$d_\Theta^x(E(x), \pi(\omega_0)) < \eta_{\omega_0}^\alpha. \quad (10)$$

or equivalently,

(C₂) the probability, that the measured value x obtained by the measurement $M_{C_0(\Omega)}(\mathbf{O} := (X, \mathcal{F}, F), S_{[\omega_0]})$ satisfies the following condition (11), is less than or equal to α (e.g., $\alpha = 0.05$).

$$d_\Theta^x(E(x), \pi(\omega_0)) \geq \eta_{\omega_0}^\alpha. \quad (11)$$

Theorem 1 [Confidence interval and statistical hypothesis testing (cf. ref. [9])]. Let $\mathbf{O} = (X, \mathcal{F}, F)$ be an observable formulated in a commutative C^* -algebra $C_0(\Omega)$. Let $E : X \rightarrow \Theta$ and $\pi : \Omega \rightarrow \Theta$ be an estimator and a quantity respectively. Let η_ω^α be as defined in the formula (9).

From the (C₁), we assert "the confidence interval method" as follows:

(D₁) [The confidence interval method]. For any $x \in X$, define

$$D_x^{1-\alpha} = \{\pi(\omega) (\in \Theta) : d_\Theta^x(E(x), \pi(\omega)) < \eta_\omega^{1-\alpha}\}. \quad (12)$$

which is called the $(1 - \alpha)$ -confidence interval. Let $x (\in X)$ be a measured value x obtained by the measurement $M_{C_0(\Omega)}(\mathbf{O} := (X, \mathcal{F}, F), S_{[\omega_0]})$. Then, the probability that $D_x^{1-\alpha} \ni \pi(\omega_0)$ is more than or equal to $1 - \alpha$.

From the (C₂), we assert "the statistical hypothesis test" as follows:

(D₂) [The statistical hypothesis test]. Assume that a state ω_0 satisfies that $\pi(\omega_0) \in H_N (\subseteq \Theta)$, where H_N is called a "null hypothesis". Put

$$\hat{R}_{H_N}^{\alpha; \Theta} = \bigcap_{\omega \in \Omega \text{ such that } \pi(\omega) \in H_N} \{E(x) (\in \Theta) : d_\Theta^x(E(x), \pi(\omega)) \geq \eta_\omega^\alpha\}. \quad (13)$$

and also

$$\hat{R}_{H_N}^{\alpha; X} = E^{-1}(\hat{R}_{H_N}^{\alpha; \Theta}) = \bigcap_{\omega \in \Omega \text{ such that } \pi(\omega) \in H_N} \{x (\in X) : d_\Theta^x(E(x), \pi(\omega)) \geq \eta_\omega^\alpha\}. \quad (14)$$

which is respectively called the (α) -rejection region of the null hypothesis H_N . Then, the probability, that the measured value $x (\in X)$ obtained by the measurement $M_{C_0(\Omega)}(\mathbf{O} := (X, \mathcal{F}, F), S_{[\omega_0]})$ (where it should be noted that $\pi(\omega_0) \in H_N$) satisfies the following condition (15), is less than or equal to α (e.g., $\alpha = 0.05$).

$$"E(x) \in \hat{R}_{H_N}^{\alpha; \Theta}" \text{ or equivalently } "x \in \hat{R}_{H_N}^{\alpha; X}" \quad (15)$$

2 ANOVA in the quantum linguistic formulation of statistics

The arguments in this section are continued from Example 1.

2.1 The simplest example; Student's t -distribution

Example 2 [Student's t -distribution (cf. [9]). Consider the measurement $M_{C_0(\mathbb{R} \times \mathbb{R}_+)} (O_G^n = (\mathbb{R}^n, \mathcal{B}_{\mathbb{R}}^n, G^n), S_{[(\mu, \sigma)]})$ in $C_0(\mathbb{R} \times \mathbb{R}_+)$ in Example 1. Thus, we consider that $\Omega = \mathbb{R} \times \mathbb{R}_+$, $X = \mathbb{R}^n$. Put $\Theta = \mathbb{R}$. Also, define the estimator $E : X (= \mathbb{R}^n) \rightarrow \Theta (= \mathbb{R})$ such that

$$E(x) = E(x_1, x_2, \dots, x_n) = \bar{\mu}(x) = \frac{x_1 + x_2 + \dots + x_n}{n} \quad (16)$$

The quantity $\pi : \Omega (= \mathbb{R} \times \mathbb{R}_+) \rightarrow \Theta (= \mathbb{R})$ is defined by

$$\Omega (= \mathbb{R} \times \mathbb{R}_+) \ni \omega = (\mu, \sigma) \mapsto \pi(\mu, \sigma) = \mu \in \Theta (= \mathbb{R}) \quad (17)$$

Also, assume that the $\Theta (= \mathbb{R})$ has the semi-distance $d_{\Theta}^x (\forall x \in X)$ such that

$$d_{\Theta}^x(\theta^{(1)}, \theta^{(2)}) = \frac{|\theta^{(1)} - \theta^{(2)}|}{\sqrt{n\bar{\sigma}(x)}} = \frac{|\theta^{(1)} - \theta^{(2)}|}{\sqrt{SS(x)}} \quad (\forall x \in X = \mathbb{R}^n, \forall \theta^{(1)}, \theta^{(2)} \in \Theta = \mathbb{R}) \quad (18)$$

where $\bar{\sigma}(x)$ is motivated by Fisher's maximum likelihood method (see the formulas (4) and (5)). Define the null hypothesis $H_N (\subseteq \Theta = \mathbb{R})$ such that

$$H_N = \{\mu_0\} \quad (19)$$

Thus, for any $\omega = (\mu_0, \sigma) (\in \Omega = \mathbb{R} \times \mathbb{R}_+)$, we see that

$$\begin{aligned} & [G^n(\{x \in X : d_{\Theta}^x(E(x), \pi(\omega)) \geq \eta\})](\omega) \\ &= [G^n(\{x \in X : \frac{|\bar{\mu}(x) - \mu_0|}{\sqrt{SS(x)}} \geq \eta\})](\omega) \\ &= \frac{1}{(\sqrt{2\pi}\sigma)^n} \int \dots \int_{\eta\sqrt{n-1} \leq \frac{|\bar{\mu}(x) - \mu_0|}{\sqrt{SS(x)}/\sqrt{n-1}}} \exp[-\frac{\sum_{k=1}^n (x_k - \mu_0)^2}{2\sigma^2}] dx_1 dx_2 \dots dx_n \\ &= \frac{1}{(\sqrt{2\pi})^n} \int \dots \int_{\eta^2 n(n-1) \leq \frac{n(\bar{\mu}(x))^2}{SS(x)/(n-1)}} \exp[-\frac{\sum_{k=1}^n (x_k)^2}{2}] dx_1 dx_2 \dots dx_n \end{aligned} \quad (20)$$

(E₁) using the formula of Gauss integrals derived in Kolmogorov's probability theory (also, see Remark 2 below), we finally get as follows.

$$= \int_{\eta^2 n(n-1)}^{\infty} p_{(1, n-1)}^F(t) dt = \alpha \quad (\text{e.g., } \alpha = 0.05) \quad (21)$$

where $p_{(1, n-1)}^F$ is the probability density function of the F -distribution with $(1, n-1)$ degrees of freedom. Recall the probability density function $p_{(n_1, n_2)}^F(x)$ of the F -distribution with (n_1, n_2) degrees of freedom is represented as follows:

$$p_{(n_1, n_2)}^F(t) = \frac{1}{B(n_1/2, n_2/2)} \left(\frac{n_1}{n_2}\right)^{n_1/2} \frac{t^{(n_1-2)/2}}{(1 + n_1 t/n_2)^{(n_1+n_2)/2}} \quad (t \geq 0) \quad (22)$$

where $B(\cdot, \cdot)$ is the Beta function. Define the α -point $F_{n_1, \alpha}^{n_2}$ (> 0) such that

$$\int_{F_{n_1, \alpha}^{n_2}}^{\infty} p_{(n_1, n_2)}^F(t) dt = \alpha \quad (0 < \alpha \ll 1. \text{ e.g., } \alpha = 0.05) \quad (23)$$

Thus, it suffices to put

$$\eta^2 n(n-1) = F_{n-1, \alpha}^1 \quad (24)$$

And thus,

$$(\eta_\omega^\alpha)^2 = \frac{F_{n-1, \alpha}^1}{n(n-1)} \quad (25)$$

Therefore, we get $\hat{R}_{H_N}^{\alpha; \Theta}$ (or $\hat{R}_{H_N}^{\alpha; X}$; the (α) -rejection region of $H_N(= \{\mu_0\})$) as follows:

$$\begin{aligned} \hat{R}_{H_N}^{\alpha; \Theta} &= \bigcap_{\omega=(\mu, \sigma) \in \Omega(=\mathbb{R} \times \mathbb{R}_+) \text{ such that } \pi(\omega)=\mu \in H_N(=\{\mu_0\})} \{E(x) \in \Theta : d_\Theta^x(E(x), \pi(\omega)) \geq \eta_\omega^\alpha\} \\ &= \{\bar{\mu}(x) \in \Theta(=\mathbb{R}) : \frac{|\bar{\mu}(x) - \mu_0|}{\sqrt{SS(x)}} \geq \eta_\omega^\alpha\} = \{\bar{\mu}(x) \in \Theta(=\mathbb{R}) : \frac{|\bar{\mu}(x) - \mu_0|}{\bar{\sigma}(x)} \geq \eta_\omega^\alpha \sqrt{n}\} \\ &= \{\bar{\mu}(x) \in \Theta(=\mathbb{R}) : \frac{|\bar{\mu}(x) - \mu_0|}{\bar{\sigma}(x)} \geq \sqrt{\frac{F_{n-1, \alpha}^1}{n-1}}\} \\ &= \{\bar{\mu}(x) \in \Theta(=\mathbb{R}) : \mu_0 \leq \bar{\mu}(x) - \bar{\sigma}(x) \sqrt{\frac{F_{n-1, \alpha}^1}{n-1}} \text{ or } \bar{\mu}(x) + \bar{\sigma}(x) \sqrt{\frac{F_{n-1, \alpha}^1}{n-1}} \leq \mu_0\} \end{aligned} \quad (26)$$

and

$$\begin{aligned} \hat{R}_{H_N}^{\alpha; X} &= E^{-1}(\hat{R}_{H_N}^{\alpha; \Theta}) \\ &= \{x \in X(=\mathbb{R}^n) : \mu_0 \leq \bar{\mu}(x) - \bar{\sigma}(x) \sqrt{\frac{F_{n-1, \alpha}^1}{n-1}} \text{ or } \bar{\mu}(x) + \bar{\sigma}(x) \sqrt{\frac{F_{n-1, \alpha}^1}{n-1}} \leq \mu_0\} \end{aligned} \quad (27)$$

Therefore, the statistical hypothesis test (D₂) in Theorem 1 is applicable.

Remark 2 [Kolmogorov's probability theory (cf. [10]). There are several derivations of (21) from (20). Of course, the formula (21) can be directly derived from the (20), though the calculation is not easy. However, Kolmogorov's probability theory is useful for the derivation of ((1). Here, let us remark it as follows. Consider the probability space $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n}, P)$, where $P(\Xi) = [G^n(\Xi)](\mu, \sigma)$ ($\forall \Xi \in \mathcal{B}_{\mathbb{R}^n}$). And, for each $k = 1, 2, \dots, n$, consider a random variable $X_k : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$X_k(x) = X_k(x_1, x_2, \dots, x_n) = x_k \quad (\forall x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n)$$

It is clear that random variables $\{X_k\}_{k=1,2,\dots,n}$ are independent with the normal distribution $N(\mu, \sigma^2)$. Define the random variables $\bar{\mu} : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\bar{\sigma} : \mathbb{R}^n \rightarrow \mathbb{R}$, which are also independent (cf. Cochran's theorem, etc.). And thus, we can easily show that the random variable $\frac{n\bar{\mu}(x)^2}{\bar{\sigma}(x)^2}$ has the F -distribution with $(1, n-1)$ degrees of freedom. Also, the random variable $\frac{\bar{\mu}(x) - \mu}{\bar{\sigma}(x)/\sqrt{n-1}}$ has the student's t -distribution. Therefore, Kolmogorov's probability theory provides a useful calculation method to quantum language. We never consider that Kolmogorov's probability theory gives a foundation to statistics. However, it is certain that mathematical theories (particularly, the theory of probability and the theory of operator algebra (cf. [12])) are indispensable for quantum language. For completeness, again note that Kolmogorov's probability theory is merely used in order to calculate multi-dimensional Gauss integrals throughout this paper (cf. the items (E₁)-(E₄) in Examples 2-5).

2.2 The one-way ANOVA

Example 3 [The one-way ANOVA]. For each $i = 1, 2, \dots, a$, a natural number n_i is determined. And put $n = \sum_{i=1}^a n_i$. As one of generalizations of Example 2, we consider a kind of normal observable $\mathbf{O}_G^n = (X(\equiv \mathbb{R}^n), \mathcal{B}_{\mathbb{R}}^n, G^n)$ in $C_0(\Omega(\equiv (\mathbb{R}^a \times \mathbb{R}_+)))$ as follows:

$$[G^n(\widehat{\Xi})](\omega) = \frac{1}{(\sqrt{2\pi}\sigma)^n} \int \cdots \int_{\widehat{\Xi}} \exp\left[-\frac{\sum_{i=1}^a \sum_{k=1}^{n_i} (x_{ik} - \mu_i)^2}{2\sigma^2}\right] \times_{i=1}^a \times_{k=1}^{n_i} dx_{ik} \quad (28)$$

$$(\forall \omega = (\mu_1, \mu_2, \dots, \mu_a, \sigma) \in \Omega = \mathbb{R}^a \times \mathbb{R}_+, \widehat{\Xi} \in \mathcal{B}_{\mathbb{R}}^n)$$

Put

$$\alpha_i = \mu_i - \frac{\sum_{i=1}^a \mu_i}{a} \quad (\forall i = 1, 2, \dots, a) \quad (29)$$

and

$$\Theta = \mathbb{R}^a \quad (30)$$

and define the map $\pi : \Omega \rightarrow \Theta$ such that

$$\Omega = \mathbb{R}^a \times \mathbb{R}_+ \ni \omega = (\mu_1, \mu_2, \dots, \mu_a, \sigma) \mapsto \pi(\omega) = (\alpha_1, \alpha_2, \dots, \alpha_a) \in \Theta = \mathbb{R}^a \quad (31)$$

Define the null hypothesis $H_N(\subseteq \Theta = \mathbb{R}^a)$ such that

$$\begin{aligned} H_N &= \{(\alpha_1, \alpha_2, \dots, \alpha_a) \in \Theta = \mathbb{R}^a : \alpha_1 = \alpha_2 = \dots = \alpha_a = \alpha\} \\ &= \{\overbrace{(0, 0, \dots, 0)}^a\} \end{aligned} \quad (32)$$

since it clearly holds that " $\mu_1 = \mu_2 = \dots = \mu_a$ " \Leftrightarrow " $\alpha_1 = \alpha_2 = \dots = \alpha_a = 0$ ".

Put

$$\begin{aligned} \|\theta^{(1)} - \theta^{(2)}\|_{\Theta} &= \sqrt{\sum_{i=1}^a n_i (\theta_i^{(1)} - \theta_i^{(2)})^2} \\ (\forall \theta^{(\ell)} &= (\theta_1^{(\ell)}, \theta_2^{(\ell)}, \dots, \theta_a^{(\ell)}) \in \mathbb{R}^a, \ell = 1, 2) \end{aligned} \quad (33)$$

which is the weighted Euclidean norm in \mathbb{R}^{n_i} .

Also, put

$$\begin{aligned} X &= \mathbb{R}^n \ni x = ((x_{ik})_{k=1,2,\dots,n_i})_{i=1,2,\dots,a} \\ x_{i\cdot} &= \frac{\sum_{k=1}^{n_i} x_{ik}}{n_i}, \quad x_{..} = \frac{\sum_{i=1}^a \sum_{k=1}^{n_i} x_{ik}}{n_i}, \end{aligned} \quad (34)$$

According to Fisher's maximum likelihood method, define and calculate $\overline{\sigma}(x) (= \sqrt{\frac{\overline{SS}(x)}{n}})$ concerning (28) as follows. For each $x \in X = \mathbb{R}^n$,

$$\begin{aligned} \overline{SS}(x) &= \overline{SS}(((x_{ik})_{k=1,2,\dots,n_i})_{i=1,2,\dots,a}) \\ &= \sum_{i=1}^a \sum_{k=1}^{n_i} (x_{ik} - x_{i\cdot})^2 \\ &= \sum_{i=1}^a \sum_{k=1}^{n_i} (x_{ik} - \frac{\sum_{k=1}^{n_i} x_{ik}}{n_i})^2 \\ &= \sum_{i=1}^a \sum_{k=1}^{n_i} ((x_{ik} - \mu_i) - \frac{\sum_{k=1}^{n_i} (x_{ik} - \mu_i)}{n_i})^2 \\ &= \overline{SS}(((x_{ik} - \mu_i)_{k=1,2,\dots,n_i})_{i=1,2,\dots,a}) \end{aligned} \quad (35)$$

And, for each $x \in X = \mathbb{R}^n$, define the semi-distance d_Θ^x in Θ such that

$$d_\Theta^x(\theta^{(1)}, \theta^{(2)}) = \frac{\|\theta^{(1)} - \theta^{(2)}\|_\Theta}{\sqrt{SS(x)}} \quad (\forall \theta^{(1)}, \theta^{(2)} \in \Theta). \quad (36)$$

Further define the estimator $E : X (= \mathbb{R}^n) \rightarrow \Theta (= \mathbb{R}^a)$ such that

$$\begin{aligned} E(x) &= E((x_{ik})_{i=1,2,\dots,a,k=1,2,\dots,n}) \\ &= \left(\frac{\sum_{k=1}^{n_1} x_{1k}}{n} - \frac{\sum_{i=1}^a \sum_{k=1}^{n_i} x_{ik}}{n}, \frac{\sum_{k=1}^{n_2} x_{2k}}{n} - \frac{\sum_{i=1}^a \sum_{k=1}^{n_i} x_{ik}}{n}, \dots, \frac{\sum_{k=1}^{n_a} x_{ak}}{n} - \frac{\sum_{i=1}^a \sum_{k=1}^{n_i} x_{ik}}{n} \right) \\ &= \left(\frac{\sum_{k=1}^{n_i} x_{ik}}{n} - \frac{\sum_{i=1}^a \sum_{k=1}^{n_i} x_{ik}}{n} \right)_{i=1,2,\dots,a} = (x_{i\cdot} - x_{\cdot\cdot})_{i=1,2,\dots,a} \end{aligned} \quad (37)$$

Hence, we see that

$$\begin{aligned} &\|E(x) - \pi(\omega)\|_\Theta^2 \\ &= \left\| \left(\frac{\sum_{k=1}^{n_i} x_{ik}}{n} - \frac{\sum_{i=1}^a \sum_{k=1}^{n_i} x_{ik}}{n} \right)_{i=1,2,\dots,a} - (\alpha_i)_{i=1,2,\dots,a} \right\|_\Theta^2 \\ &= \left\| \left(\frac{\sum_{k=1}^{n_i} x_{ik}}{n} - \frac{\sum_{i=1}^a \sum_{k=1}^{n_i} x_{ik}}{n} - \left(\mu_i - \frac{\sum_{i=1}^a \mu_i}{a} \right) \right)_{i=1,2,\dots,a} \right\|_\Theta^2 \end{aligned} \quad (38)$$

and thus, if the null hypothesis H_N is assumed (i.e., $\mu_i - \frac{\sum_{i=1}^a \mu_i}{a} = \alpha_i = 0 (i = 1, 2, \dots, a)$), we see

$$= \left\| \left(\frac{\sum_{k=1}^{n_i} x_{ik}}{n} - \frac{\sum_{i=1}^a \sum_{k=1}^{n_i} x_{ik}}{n} \right)_{i=1,2,\dots,a} \right\|_\Theta^2 = \sum_{i=1}^a n_i (x_{i\cdot} - x_{\cdot\cdot})^2 \quad (39)$$

Thus, for any $\omega = ((\mu_{ik})_{i=1,2,\dots,a,k=1,2,\dots,n}, \sigma) (\in \Omega = \mathbb{R}^n \times \mathbb{R}_+)$, define the positive number $\eta_\omega^\alpha (> 0)$ such that:

$$\eta_\omega^\alpha = \inf\{\eta > 0 : [G^n(E^{-1}(\text{Ball}_{d_\Theta^C}^C(\pi(\omega); \eta)))(\omega) \geq \alpha]\} \quad (40)$$

where

$$\text{Ball}_{d_\Theta^C}^C(\pi(\omega); \eta) = \{\theta \in \Theta : d_\Theta^x(\pi(\omega), \theta) > \eta\} \quad (41)$$

Assume the null hypothesis H_N (i.e., $\mu_i - \frac{\sum_{i=1}^a \mu_i}{a} = \alpha_i = 0 (i = 1, 2, \dots, a)$). Now let us calculate the η_ω^α as follows:

$$\begin{aligned} E^{-1}(\text{Ball}_{d_\Theta^C}^C(\pi(\omega); \eta)) &= \{x \in X = \mathbb{R}^n : d_\Theta^x(E(x), \pi(\omega)) > \eta\} \\ &= \{x \in X = \mathbb{R}^n : \frac{\|E(x) - \pi(\omega)\|_\Theta^2}{SS(x)} = \frac{\sum_{i=1}^a n_i (x_{i\cdot} - x_{\cdot\cdot})^2}{\sum_{i=1}^a \sum_{k=1}^{n_i} (x_{ik} - x_{i\cdot})^2} > \eta^2\} \end{aligned} \quad (42)$$

That is, for any $\omega = (\mu_1, \mu_2, \dots, \mu_a, \sigma) \in \Omega = \mathbb{R}^a \times \mathbb{R}_+$ such that $\pi(\omega) = (\alpha_1, \alpha_2, \dots, \alpha_a) \in H_N (= \{0, 0, \dots, 0\})$,

$$\begin{aligned} &[G^n(E^{-1}(\text{Ball}_{d_\Theta^C}^C(\pi(\omega); \eta)))(\omega) \\ &= \frac{1}{(\sqrt{2\pi}\sigma)^n} \int \cdots \int_{\frac{\sum_{i=1}^a n_i (x_{i\cdot} - x_{\cdot\cdot})^2}{\sum_{i=1}^a \sum_{k=1}^{n_i} (x_{ik} - x_{i\cdot})^2} > \eta^2} \exp\left[-\frac{\sum_{i=1}^a \sum_{k=1}^{n_i} (x_{ik} - \mu_i)^2}{2\sigma^2}\right] \times_{i=1}^a \times_{k=1}^{n_i} dx_{ik} \\ &= \frac{1}{(\sqrt{2\pi})^n} \int \cdots \int_{\frac{(\sum_{i=1}^a n_i (x_{i\cdot} - x_{\cdot\cdot})^2 / (a-1))}{(\sum_{i=1}^a \sum_{k=1}^{n_i} (x_{ik} - x_{i\cdot})^2) / (n-a)} > \frac{\eta^2 (n-a)}{(a-1)}} \exp\left[-\frac{\sum_{i=1}^a \sum_{k=1}^{n_i} (x_{ik})^2}{2}\right] \times_{i=1}^a \times_{k=1}^{n_i} dx_{ik} \end{aligned} \quad (43)$$

(E₂) using the formula of Gauss integrals derived in Kolmogorov's probability theory (also, recall Remark 2), we finally get as follows.

$$= \int_{\frac{\eta^2(n-a)}{(a-1)}}^{\infty} p_{(a-1, n-a)}^F(t) dt = \alpha \quad (\text{e.g., } \alpha=0.05) \quad (44)$$

where $p_{(a-1, n-a)}^F$ is the probability density function the F -distribution with $(a-1, n-a)$ degrees of freedom. Thus, it suffices to put

$$\frac{\eta^2(n-a)}{(a-1)} = F_{n-a, \alpha}^{a-1} (= \text{"}\alpha\text{-point"} \text{"}) \quad (45)$$

And thus we see,

$$(\eta_{\omega}^{\alpha})^2 = F_{n-a, \alpha}^{a-1} (a-1)/(n-a) \quad (46)$$

Therefore, we get $\hat{R}_{\hat{x}}^{\alpha; \Theta}$ (or, $\hat{R}_{\hat{x}}^{\alpha; X}$; the (α) -rejection region of $H_N = \{(0.0, \dots, 0)\} (\subseteq \Theta = \mathbb{R}^a)$) as follows:

$$\begin{aligned} \hat{R}_{H_N}^{\alpha; \Theta} &= \bigcap_{\omega = ((\mu_i)_{i=1}^a, \sigma) \in \Omega (= \mathbb{R}^a \times \mathbb{R}_+) \text{ such that } \pi(\omega) = (\mu)_{i=1}^a \in H_N = \{(0, 0, \dots, 0)\}} \{E(x) (\in \Theta) : d_{\Theta}^x(E(x), \pi(\omega)) \geq \eta_{\omega}^{\alpha}\} \\ &= \{E(x) (\in \Theta) : \frac{(\sum_{i=1}^a n_i (x_{i\cdot} - x_{\cdot\cdot})^2)/(a-1)}{(\sum_{i=1}^a \sum_{k=1}^{a_i} (x_{ik} - x_{i\cdot})^2)/(n-a)} \geq F_{n-a, \alpha}^{a-1}\} \end{aligned} \quad (47)$$

Thus,

$$\hat{R}_{\hat{x}}^{\alpha; X} = E^{-1}(\hat{R}_{H_N}^{\alpha; \Theta}) = \{x \in X : \frac{(\sum_{i=1}^a n_i (x_{i\cdot} - x_{\cdot\cdot})^2)/(a-1)}{(\sum_{i=1}^a \sum_{k=1}^{n_i} (x_{ik} - x_{i\cdot})^2)/(n-a)} \geq F_{n-a, \alpha}^{a-1}\} \quad (48)$$

Therefore, the statistical hypothesis test (D₂) in Theorem 1 is applicable.

2.3 The two-way ANOVA

As one of generalizations of Example 2, we consider a kind of observable $O_G^{abn} = (X (\equiv \mathbb{R}^{abn}), \mathcal{B}_{\mathbb{R}}^{abn}, G^{abn})$ in $C_0(\Omega (\equiv (\mathbb{R}^{ab} \times \mathbb{R}_+)))$.

Put

$$\begin{aligned} &[G^{abn}(\hat{\Xi})](\omega) \\ &= \frac{1}{(\sqrt{2\pi}\sigma)^{abn}} \int \cdots \int_{\hat{\Xi}} \exp\left[-\frac{\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^{n_{ij}} (x_{ijk} - \mu_{ij})^2}{2\sigma^2}\right] \times_{k=1}^n \times_{j=1}^b \times_{i=1}^a dx_{ijk} \\ &(\forall \omega = ((\mu_{ij})_{i=1,2,\dots,a,j=1,2,\dots,b}, \sigma) \in \Omega = \mathbb{R}^{ab+1} \times \mathbb{R}_+, \hat{\Xi} \in \mathcal{B}_{\mathbb{R}}^{abn}) \end{aligned} \quad (49)$$

Put

$$\begin{aligned} \mu_{ij} &= \bar{\mu} (= \mu_{\cdot\cdot} = \frac{\sum_{i=1}^a \sum_{j=1}^b \mu_{ij}}{ab}) \\ &+ \alpha_i (= \mu_{i\cdot} - \mu_{\cdot\cdot} = \frac{\sum_{j=1}^b \mu_{ij}}{b} - \frac{\sum_{i=1}^a \sum_{j=1}^b \mu_{ij}}{ab}) \\ &+ \beta_j (= \mu_{\cdot j} - \mu_{\cdot\cdot} = \frac{\sum_{i=1}^a \mu_{ij}}{a} - \frac{\sum_{i=1}^a \sum_{j=1}^b \mu_{ij}}{ab}) \\ &+ (\alpha\beta)_{ij} (= \mu_{ij} - \mu_{i\cdot} - \mu_{\cdot j} + \mu_{\cdot\cdot}) \end{aligned} \quad (50)$$

Put

$$\begin{aligned}
X &= \mathbb{R}^{abn} \ni x = (x_{ijk})_{i=1,2,\dots,a, j=1,2,\dots,b, k=1,2,\dots,n} \\
x_{ij\cdot} &= \frac{\sum_{k=1}^n x_{ijk}}{n}, \quad x_{i\cdot\cdot} = \frac{\sum_{j=1}^b \sum_{k=1}^n x_{ijk}}{bn}, \quad x_{\cdot j\cdot} = \frac{\sum_{i=1}^a \sum_{k=1}^n x_{ijk}}{an}, \\
x_{\dots} &= \frac{\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n x_{ijk}}{abn}
\end{aligned} \tag{51}$$

Example 4 [The null hypothesis such that $\alpha_1 = \alpha_2 = \dots = \alpha_a = 0$]. Define the $\pi : \Omega \rightarrow \Theta$ such that

$$\Omega = \mathbb{R}^{ab+1} \times \mathbb{R}_+ \ni \omega = ((\mu_{ij})_{i=1,2,\dots,a, j=1,2,\dots,b}, \sigma) \mapsto \pi_1(\omega) = (\alpha_i)_{i=1}^a \in \Theta = \mathbb{R}^a \tag{52}$$

Put

$$\Theta = \mathbb{R}^a \tag{53}$$

and define the $\pi : \Omega \rightarrow \Theta$ such that

$$\Omega = \mathbb{R}^a \times \mathbb{R}_+ \ni \omega = (\mu_1, \mu_2, \dots, \mu_a, \sigma) \mapsto \pi(\omega) = (\alpha_1, \alpha_2, \dots, \alpha_a) \in \Theta = \mathbb{R}^a \tag{54}$$

Define the null hypothesis $H_N (\subseteq \Theta = \mathbb{R}^a)$ such that

$$\begin{aligned}
H_N &= \{(\alpha_1, \alpha_2, \dots, \alpha_a) \in \Theta = \mathbb{R}^a : \alpha_1 = \alpha_2 = \dots = \alpha_a = \alpha\} \\
&= \{\overbrace{(0, 0, \dots, 0)}^a\}
\end{aligned} \tag{55}$$

That is because

$$a\alpha = \sum_{i=1}^a \alpha_i = \sum_{i=1}^a (\mu_{i\cdot} - \mu_{\cdot\cdot}) = \frac{\sum_{i=1}^a \sum_{j=1}^b \mu_{ij}}{b} - \sum_{i=1}^a \frac{\sum_{i=1}^a \sum_{j=1}^b \mu_{ij}}{ab} = 0 \tag{56}$$

Put

$$\begin{aligned}
\|\theta^{(1)} - \theta^{(2)}\|_{\Theta} &= \sqrt{\sum_{i=1}^a (\theta_i^{(1)} - \theta_i^{(2)})^2} \\
(\forall \theta^{(\ell)} &= (\theta_1^{(\ell)}, \theta_2^{(\ell)}, \dots, \theta_a^{(\ell)}) \in \mathbb{R}^a, \ell = 1, 2)
\end{aligned} \tag{57}$$

Motivated by Fisher's maximum likelihood method, define and calculate $\bar{\sigma}(x) \left(= \sqrt{\overline{SS}(x)/(abn)} \right)$ as follows.

$$\begin{aligned}
\overline{SS}(x) &= \overline{SS}((x_{ijk})_{i=1,2,\dots,a, j=1,2,\dots,b, k=1,2,\dots,n}) \\
&:= \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (x_{ijk} - x_{ij\cdot})^2 = \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n \left(x_{ijk} - \frac{\sum_{k=1}^n x_{ijk}}{n} \right)^2 \\
&= \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n \left((x_{ijk} - \mu_{ij}) - \frac{\sum_{k=1}^n (x_{ijk} - \mu_{ij})}{n} \right)^2 \\
&= \overline{SS}((x_{ijk} - \mu_{ij})_{i=1,2,\dots,a, j=1,2,\dots,b, k=1,2,\dots,n})
\end{aligned} \tag{58}$$

Define the semi-distance d_{Θ}^x in $\Theta = \mathbb{R}^a$ such that

$$d_{\Theta}^x(\theta^{(1)}, \theta^{(2)}) = \frac{\|\theta^{(1)} - \theta^{(2)}\|_{\Theta}}{\sqrt{\overline{SS}(x)}} \quad (\forall \theta^{(1)}, \theta^{(2)} \in \Theta = \mathbb{R}^a, \forall x \in X = \mathbb{R}^{abn}) \tag{59}$$

Define the estimator $E : X (= \mathbb{R}^{abn}) \rightarrow \Theta (= \mathbb{R}^a)$ such that

$$E(x) = \left(\frac{\sum_{j=1}^b \sum_{k=1}^n x_{ijk}}{bn} - \frac{\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n x_{ijk}}{abn} \right)_{i=1,2,\dots,a} = (x_{i\cdot\cdot} - x_{\dots})_{i=1,2,\dots,a} \quad (60)$$

Hence

$$\begin{aligned} & \|E(x) - \pi(\omega)\|_{\Theta}^2 \\ &= \left\| \left(\frac{\sum_{j=1}^b \sum_{k=1}^n x_{ijk}}{bn} - \frac{\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n x_{ijk}}{abn} \right)_{i=1,2,\dots,a} - (\alpha_i)_{i=1,2,\dots,a} \right\|_{\Theta}^2 \\ &= \left\| \left(\frac{\sum_{j=1}^b \sum_{k=1}^n x_{ijk}}{bn} - \frac{\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n x_{ijk}}{abn} \right)_{i=1,2,\dots,a} - \left(\frac{\sum_{j=1}^b \mu_{ij}}{b} - \frac{\sum_{i=1}^a \sum_{j=1}^b \mu_{ij}}{ab} \right)_{i=1,2,\dots,a} \right\|_{\Theta}^2 \\ &= \left\| \left(\frac{\sum_{k=1}^n \sum_{j=1}^b (x_{ijk} - \mu_{ij})}{bn} - \frac{\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (x_{ijk} - \mu_{ij})}{abn} \right)_{i=1,2,\dots,a} \right\|_{\Theta}^2 \end{aligned} \quad (61)$$

and thus, if the null hypothesis H_N is assumed (i.e., $\mu_{i\cdot} - \mu_{\cdot\cdot} = \alpha_i = 0$ ($\forall i = 1, 2, \dots, a$))

$$= \left\| \left(\frac{\sum_{k=1}^n \sum_{j=1}^b x_{ijk}}{bn} - \frac{\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n x_{ijk}}{abn} \right)_{i=1,2,\dots,a} \right\|_{\Theta}^2 = \sum_{i=1}^a (x_{ij\cdot} - x_{\dots})^2 \quad (62)$$

Thus, for any $\omega = (\mu_1, \mu_2) \in \Omega = \mathbb{R} \times \mathbb{R}$, define the positive number η_{ω}^{α} (> 0) such that:

$$\eta_{\omega}^{\alpha} = \inf\{\eta > 0 : [G(E^{-1}(\text{Ball}_{d_{\Theta}^C}(\pi(\omega); \eta)))(\omega) \geq \alpha]\} \quad (63)$$

Assume the null hypothesis H_N . Now let us calculate the η_{ω}^{α} as follows:

$$\begin{aligned} E^{-1}(\text{Ball}_{d_{\Theta}^C}(\pi(\omega); \eta)) &= \{x \in X = \mathbb{R}^{abn} : d_{\Theta}^x(E(x), \pi(\omega)) > \eta\} \\ &= \{x \in X = \mathbb{R}^{abn} : \frac{abn \sum_{i=1}^a \sum_{j=1}^b (x_{ij\cdot} - x_{\dots})^2}{\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (x_{ijk} - x_{ij\cdot})^2} > \eta\} \end{aligned} \quad (64)$$

That is, for any $\omega = ((\mu_{ij})_{i=1,2,\dots,a, j=1,2,\dots,b}, \sigma) \in \Omega$ such that $\pi(\omega) = (\alpha_1, \alpha_2, \dots, \alpha_a) \in H_N (= \{0, 0, \dots, 0\})$,

$$\begin{aligned} & [G^{abn}(E^{-1}(\text{Ball}_{d_{\Theta}^C}(\pi(\omega); \eta)))(\omega) \\ &= \frac{1}{(\sqrt{2\pi}\sigma)^{abn}} \int \cdots \int_{E^{-1}(\text{Ball}_{d_{\Theta}^C}(\pi(\omega); \eta))} \exp\left[-\frac{\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (x_{ijk} - \mu_{ij})^2}{2\sigma^2}\right] \times_{k=1}^n \times_{j=1}^b \times_{i=1}^a dx_{ijk} \\ &= \frac{1}{(\sqrt{2\pi}\sigma)^{abn}} \int \cdots \int_{\frac{abn \sum_{i=1}^a \sum_{j=1}^b (x_{ij\cdot} - x_{\dots})^2}{\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (x_{ijk} - x_{ij\cdot})^2} > \eta^2} \exp\left[-\frac{\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (x_{ijk} - \mu_{ij})^2}{2\sigma^2}\right] \times_{k=1}^n \times_{j=1}^b \times_{i=1}^a dx_{ijk} \\ &= \frac{1}{(\sqrt{2\pi})^{abn}} \int \cdots \int_{\frac{\sum_{i=1}^a \sum_{j=1}^b (x_{ij\cdot} - x_{\dots})^2}{(a-1)} > \frac{\eta^2(ab(n-1))}{abn(a-1)}} \exp\left[-\frac{\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (x_{ijk})^2}{2}\right] \times_{k=1}^n \times_{j=1}^b \times_{i=1}^a dx_{ijk} \end{aligned} \quad (65)$$

(E₃) using the formula of Gauss integrals derived in Kolmogorov's probability theory (also, recall Remark 2), we finally get as follows.

$$= \int_{\frac{\eta^2(n-1)}{n(a-1)}}^{\infty} p_{(a-1, ab(n-1))}^F(t) dt = \alpha \text{ (e.g., } \alpha = 0.05) \quad (66)$$

where $p_{(a-1, ab(n-1))}^F$ is the F -distribution with $(a-1, ab(n-1))$ degrees of freedom. Thus, as seen in the formula (46), it suffices to calculate the α -point $F_{ab(n-1), \alpha}^{a-1}$. Thus, we see

$$(\eta_\omega^\alpha)^2 = F_{ab(n-1), \alpha}^{a-1} \cdot n(a-1)/(n-1) \quad (67)$$

Therefore, we get $\hat{R}_{\hat{x}}^{\alpha; \Theta}$ (or, $\hat{R}_{\hat{x}}^{\alpha; X}$; the (α) -rejection region of $H_N = \{(0.0, \dots, 0)\} (\subseteq \Theta = \mathbb{R}^a)$) as follows:

$$\begin{aligned} \hat{R}_{H_N}^{\alpha; \Theta} &= \bigcap_{\omega = ((\mu_i)_{i=1}^a, \sigma) \in \Omega (= \mathbb{R}^a \times \mathbb{R}_+) \text{ such that } \pi(\omega) = (\alpha_i)_{i=1}^a \in H_N = \{(0, 0, \dots, 0)\}} \{E(x) (\in \Theta) : d_\Theta^x(E(x), \pi(\omega)) \geq \eta_\omega^\alpha\} \\ &= \{E(x) (\in \Theta) : \frac{(\sum_{i=1}^a \sum_{j=1}^b (x_{ij\cdot} - x_{\dots})^2)/(a-1)}{(\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (x_{ijk} - x_{ij\cdot})^2)/(ab(n-1))} \geq F_{ab(n-1), \alpha}^{a-1}\} \end{aligned} \quad (68)$$

Thus,

$$\hat{R}_{H_N}^{\alpha; X} = E^{-1}(\hat{R}_{H_N}^{\alpha; \Theta}) = \{x (\in X) : \frac{(\sum_{i=1}^a \sum_{j=1}^b (x_{ij\cdot} - x_{\dots})^2)/(a-1)}{(\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (x_{ijk} - x_{ij\cdot})^2)/(ab(n-1))} \geq F_{ab(n-1), \alpha}^{a-1}\} \quad (69)$$

Therefore, the statistical hypothesis test (D₂) in Theorem 1 is applicable.

Remark 3 If we assume the null hypothesis such that $\beta_1 = \beta_2 = \dots = \beta_b = 0$, we can give the similar answer such as Example 4.

Example 5 [The null hypothesis such that $(\alpha\beta)_{ij} = 0$ ($\forall i = 1, 2, \dots, a, j = 1, 2, \dots, b$). Put

$$\Theta = \mathbb{R}^{ab} \quad (70)$$

and define the $\pi : \Omega \rightarrow \Theta$ such that

$$\Omega = \mathbb{R}^a \times \mathbb{R}_+ \ni \omega = (\mu_1, \mu_2, \dots, \mu_a, \sigma) \mapsto \pi(\omega) = ((\alpha\beta)_{ij})_{i=1,2,\dots,a, j=1,2,\dots,b} \in \Theta = \mathbb{R}^{ab} \quad (71)$$

where, as defined in (50),

$$(\alpha\beta)_{ij} = \mu_{ij} - \mu_{i\cdot} - \mu_{\cdot j} + \mu_{\cdot\cdot} \quad (72)$$

Define the null hypothesis $H_N (\subseteq \Theta = \mathbb{R}^{ab})$ such that

$$H_N = \{((\alpha\beta)_{ij})_{i=1,2,\dots,a, j=1,2,\dots,b} \in \Theta = \mathbb{R}^{ab} : (\alpha\beta)_{ij} = 0, (\forall i = 1, 2, \dots, a, j = 1, 2, \dots, b)\} \quad (73)$$

Put

$$\begin{aligned} \|\theta^{(1)} - \theta^{(2)}\|_\Theta &= \sqrt{\sum_{i=1}^a \sum_{j=1}^b (\theta_{ij}^{(\ell)} - \theta_{ij}^{(\ell)})^2} \\ (\forall \theta^{(\ell)} &= (\theta_{ij}^{(\ell)})_{i=1,2,\dots,a, j=1,2,\dots,b} \in \mathbb{R}^{ab}, \ell = 1, 2) \end{aligned} \quad (74)$$

Define $\overline{SS}(x)$ by the formula (58), and define the semi-distance d_Θ^x in Θ such that

$$d_\Theta^x(\theta^{(1)}, \theta^{(2)}) = \frac{\|\theta^{(1)} - \theta^{(2)}\|_\Theta}{\sqrt{\overline{SS}(x)}} \quad (\forall \theta^{(1)}, \theta^{(2)} \in \Theta, \forall x \in X) \quad (75)$$

Define and calculate the estimator $E : X(= \mathbb{R}^{abn}) \rightarrow \Theta(= \mathbb{R}^{ab})$ such that

$$\begin{aligned} & E((x_{ijk})_{i=1,\dots,a, j=1,2,\dots,b, k=1,2,\dots,n}) \\ &= \left(\frac{\sum_{k=1}^n x_{ijk}}{n} - \frac{\sum_{j=1}^b \sum_{k=1}^n x_{ijk}}{bn} - \frac{\sum_{j=1}^b \sum_{k=1}^n x_{ijk}}{an} + \frac{\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n x_{ijk}}{abn} \right)_{i=1,2,\dots,a, j=1,2,\dots,b,} \\ &= \left(x_{ij\cdot} - x_{i\cdot\cdot} - x_{\cdot j\cdot} + x_{\dots} \right)_{i=1,2,\dots,a, j=1,2,\dots,b,} \end{aligned} \quad (76)$$

and thus,

$$\begin{aligned} & E((x_{ijk} - \mu_{ij})_{i=1,\dots,a, j=1,2,\dots,b, k=1,2,\dots,n}) \\ &= \left(\frac{\sum_{k=1}^n (x_{ijk} - \mu_{ij})}{n} - \frac{\sum_{j=1}^b \sum_{k=1}^n (x_{ijk} - \mu_{ij})}{bn} - \frac{\sum_{j=1}^b \sum_{k=1}^n (x_{ijk} - \mu_{ij})}{an} + \frac{\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (x_{ijk} - \mu_{ij})}{abn} \right)_{i=1,2,\dots,a, j=1,2,\dots,b,} \\ &= \left((x_{ij\cdot} - \mu_{ij}) - (x_{i\cdot\cdot} - \mu_{i\cdot\cdot}) - (x_{\cdot j\cdot} - \mu_{\cdot j\cdot}) + (x_{\dots} - \mu_{\dots}) \right)_{i=1,2,\dots,a, j=1,2,\dots,b,} \\ &= \left(x_{ij\cdot} - x_{i\cdot\cdot} - x_{\cdot j\cdot} + x_{\dots} \right)_{i=1,2,\dots,a, j=1,2,\dots,b,} \quad (\text{under the null hypothesis } (\alpha\beta)_{ij} = 0) \end{aligned} \quad (77)$$

Therefore,

$$E((x_{ijk})_{i=1,\dots,a, j=1,2,\dots,b, k=1,2,\dots,n}) = E((x_{ijk} - \mu_{ij})_{i=1,\dots,a, j=1,2,\dots,b, k=1,2,\dots,n}) \quad (78)$$

Hence, for each $i = 1, \dots, a, j = 1, 2, \dots, b$,

$$\begin{aligned} & E_{ij}(x_{ijk} - \mu_{ij}) \\ &= \frac{\sum_{k=1}^n (x_{ijk} - \mu_{ij})}{n} - \frac{\sum_{j=1}^b \sum_{k=1}^n (x_{ijk} - \mu_{ij})}{bn} - \frac{\sum_{j=1}^b \sum_{k=1}^n (x_{ijk} - \mu_{ij})}{an} \\ & \quad + \frac{\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (x_{ijk} - \mu_{ij})}{abn} \\ &= E_{ij}(x) - (\alpha\beta)_{ij} \\ &= x_{ij\cdot} - x_{i\cdot\cdot} - x_{\cdot j\cdot} + x_{\dots} - (\alpha\beta)_{ij} \end{aligned} \quad (79)$$

Thus, we see that

$$\begin{aligned} & \|E(x) - \pi(\omega)\|_{\Theta}^2 \\ &= \left\| \left(E_{ij}(x) - (\alpha\beta)_{ij} \right)_{i=1,2,\dots,a, j=1,2,\dots,b} \right\|_{\Theta}^2 \end{aligned} \quad (80)$$

and thus, if the null hypothesis H_N is assumed (i.e., $(\alpha\beta)_{ij} = 0$ ($\forall i = 1, 2, \dots, a, j = 1, 2, \dots, b$))

$$= \sum_{i=1}^a \sum_{j=1}^b (x_{ij\cdot} - x_{i\cdot\cdot} - x_{\cdot j\cdot} + x_{\dots})^2 \quad (81)$$

Thus, for any $\omega = (\mu_1, \mu_2) \in \Omega = \mathbb{R} \times \mathbb{R}$, define the positive number η_{ω}^{α} (> 0) such that:

$$\eta_{\omega}^{\alpha} = \inf\{\eta > 0 : [G(E^{-1}(\text{Ball}_{d_{\Theta}^C}(\pi(\omega); \eta)))(\omega) \geq \alpha]\} \quad (82)$$

Assume the null hypothesis H_N (i.e., $(\alpha\beta)_{ij} = 0$ ($\forall i = 1, 2, \dots, a, j = 1, 2, \dots, b$)). Now let us calculate the η_ω^α as follows:

$$\begin{aligned} E^{-1}(\text{Ball}_{d_{\Theta}^x}^C(\pi(\omega); \eta)) &= \{x \in X = \mathbb{R}^{abn} : d_{\Theta}^x(E(x), \pi(\omega)) > \eta\} \\ &= \{x \in X = \mathbb{R}^{abn} : \frac{abn \sum_{i=1}^a \sum_{j=1}^b (x_{ij\cdot} - x_{i\cdot\cdot} - x_{\cdot j\cdot} + x_{\cdot\cdot\cdot})^2}{\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (x_{ijk} - x_{ij\cdot})^2} > \eta^2\} \end{aligned} \quad (83)$$

That is, for any $\omega = ((\mu_{ij})_{i=1,2,\dots,a, j=1,2,\dots,b}, \sigma) \in \Omega = \mathbb{R}^{ab+1}$ such tht $\pi(\omega) \in H_N (\subseteq \mathbb{R}^{ab})$ (i.e., $(\alpha\beta)_{ij} = 0$ ($\forall i = 1, 2, \dots, a, j = 1, 2, \dots, b$))

$$\begin{aligned} &[G^{abn}(E^{-1}(\text{Ball}_{d_{\Theta}^x}^C(\pi(\omega); \eta)))(\omega) \\ &= \frac{1}{(\sqrt{2\pi}\sigma)^{abn}} \int \cdots \int_{E^{-1}(\text{Ball}_{d_{\Theta}^x}^C(\pi(\omega); \eta))} \exp\left[-\frac{\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (x_{ijk} - \mu_{ij})^2}{2\sigma^2}\right] \times \prod_{k=1}^n \prod_{j=1}^b \prod_{i=1}^a dx_{ijk} \\ &= \frac{1}{(\sqrt{2\pi}\sigma)^{abn}} \int \cdots \int_{\{x \in X : d_{\Theta}^x(E(x), \pi(\omega)) \geq \eta\}} \exp\left[-\frac{\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (x_{ijk} - \mu_{ij})^2}{2\sigma^2}\right] \times \prod_{k=1}^n \prod_{j=1}^b \prod_{i=1}^a dx_{ijk} \\ &= \frac{1}{(\sqrt{2\pi})^{abn}} \int \cdots \int \exp\left[-\frac{\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (x_{ijk})^2}{2}\right] \times \prod_{k=1}^n \prod_{j=1}^b \prod_{i=1}^a dx_{ijk} \\ &\quad \frac{\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (x_{ij\cdot} - x_{i\cdot\cdot} - x_{\cdot j\cdot} + x_{\cdot\cdot\cdot})^2}{\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (x_{ijk} - x_{ij\cdot})^2} > \frac{\eta^2}{abn} \\ &= \frac{1}{(\sqrt{2\pi})^{abn}} \int \cdots \int \exp\left[-\frac{\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (x_{ijk})^2}{2}\right] \times \prod_{k=1}^n \prod_{j=1}^b \prod_{i=1}^a dx_{ijk} \\ &\quad \frac{\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (x_{ij\cdot} - x_{i\cdot\cdot} - x_{\cdot j\cdot} + x_{\cdot\cdot\cdot})^2}{\frac{(a-1)(b-1)}{\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (x_{ijk} - x_{ij\cdot})^2}} > \frac{\eta^2(ab(n-1))}{abn(a-1)(b-1)} \end{aligned} \quad (84)$$

(E₄) using the formula of Gauss integrals derived in Kolmogorov's probability theory (also, recall Remark 2), we finally get as follows.

$$= \int_{\frac{\eta^2(n-1)}{n(a-1)(b-1)}}^{\infty} p_{((a-1)(b-1), ab(n-1))}^F(t) dt = \alpha \text{ (e.g., } \alpha = 0.05) \quad (85)$$

where $p_{((a-1)(b-1), ab(n-1))}^F$ is the F -distribution with $((a-1)(b-1), ab(n-1))$ degrees of freedom. Thus, as seen in the formula (67), Thus, it suffices to put

$$\frac{\eta^2(n-1)}{n(a-1)(b-1)} = F_{ab(n-1), \alpha}^{(a-1)(b-1)} (= \text{"}\alpha\text{-point"} \text{")} \quad (86)$$

And thus we see,

$$(\eta_\omega^\alpha)^2 = F_{ab(n-1), \alpha}^{(a-1)(b-1)} n(a-1)(b-1)/(n-1) \quad (87)$$

Therefore, we get $\hat{R}_{\hat{x}}^{\alpha; \Theta}$ (or, $\hat{R}_{\hat{x}}^{\alpha; X}$; the (α) -rejection region of $H_N = \{((\alpha\beta)_{ij})_{i=1,2,\dots,a, j=1,2,\dots,b} : (\alpha\beta)_{ij} = 0 \text{ (} i = 1, 2, \dots, a, j = 1, 2, \dots, b \text{)}\} (\subseteq \Theta = \mathbb{R}^{ab})$) as follows:

$$\begin{aligned} \hat{R}_{H_N}^{\alpha; \Theta} &= \bigcap_{\omega = ((\mu_{ij})_{i=1,2,\dots,a, j=1,2,\dots,b}, \sigma) \in \Omega (= \mathbb{R}^a \times \mathbb{R}_+) \text{ such that } \pi(\omega) = (\alpha\beta)_{ij} \in H_N} \{E(x) \in \Theta : d_{\Theta}^x(E(x), \pi(\omega)) \geq \eta_\omega^\alpha\} \\ &= \{E(x) \in \Theta : \frac{(\sum_{i=1}^a \sum_{j=1}^b (x_{ij\cdot} - x_{\cdot\cdot\cdot})^2)/((a-1)(b-1))}{(\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (x_{ijk} - x_{ij\cdot})^2)/(ab(n-1))} \geq F_{ab(n-1), \alpha}^{(a-1)(b-1)}\} \end{aligned} \quad (88)$$

Thus,

$$\widehat{R}_{H_N}^{\alpha;X} = E^{-1}(\widehat{R}_{H_N}^{\alpha;\Theta}) = \{x(\in X) : \frac{(\sum_{i=1}^a \sum_{j=1}^b (x_{ij\cdot} - x_{\dots})^2)/((a-1)(b-1))}{(\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (x_{ijk} - x_{ij\cdot})^2)/(ab(n-1))} \geq F_{ab(n-1),\alpha}^{(a-1)(b-1)}\} \quad (89)$$

Therefore, the statistical hypothesis test (D₂) in Theorem 1 is applicable.

3 Conclusions

We believe that quantum language has a great power of description, and therefore, even statistics can be described by quantum language. Since quantum language is suited for theoretical arguments, we believe, from the theoretical point of view, that our results (i.e., ANOVA in Section 2) are visible and simple. Therefore, we can easily answer the following question:

(F₁) Where is Kolmogorov's probability theory used in ANOVA?

As the conclusion, we can answer as follows:

(F₂) Kolmogorov's probability theory is merely used in order to calculate multi-dimensional Gauss integrals throughout this paper (*cf.* the items (E₁)-(E₄) in Examples 2-5).

It is reasonable, since Kolmogorov's probability theory is mathematics. Although we may calculate the multi-dimensional Gauss integrals without Kolmogorov's probability theory (*cf.* Remark 2), it is sure that the conventional calculation (due to Kolmogorov's probability theory) is elegant and powerful. In this sense, we believe that mathematical theories (particularly, Kolmogorov's probability theory and the theory of operator algebra (*cf.* [12])) are indispensable for quantum language.

We hope that our assertions will be examined from various points of view.

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